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2+1 dimensional charged black hole with (anti-)self dual Maxwell fields

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ABSTRACT

We discuss the exact electrically charged BTZ black hole solutions to the Einstein-Maxwell equations with a negative cosmological constant in 2+1 spacetime dimensions assuming a (anti-)self dual condition between the electromagnetic fields. In a coordinate condition there appears a logarithmic divergence in the angular momentum at spatial infinity. We show how it is to be regularized by taking the contribution from the boundary into account. We show another coordinate condition which leads to a finite angular momentum though it brings about a peculiar spacetime topology.

keywords: black hole, $2 + 1$ dimensions, self dual

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The (2+1)-dimensional black hole solution with a negative cosmological constant, which corresponds to the Kerr black hole solution in (3+1)-dimensional general relativity, was found by Banãdos, Teitelboim and Zanelli (BTZ) [1, 2]. It has mass and angular momentum. Then an exact electrically charged BTZ solution to the Einstein-Maxwell equations with a negative cosmological constant in 2+1 spacetime dimensions assuming a self dual(SD) (anti-self dual(ASD)) equation was discussed [3, 4]. It is the solution with mass M and angular momentum J which were determined by the asymptotic form of the metric. However the angular momentum has a logarithmic divergence at spatial infinity and so the solution should not be called the black hole solution [5].

In this paper we clarify the origin of the divergence and how we can circumvent the problem. In order to discuss the conserved charges we should pay attention to the definitions of them. When there is nonzero cosmological constant, we can not justify the naive definition of mass and angular momentum because they depend on the normalization of Killing vector fields, which should be contrasted with the asymptotically flat case with zero cosmological constant. We should take the contribution from the boundary into account and, in this case, we have an arbitrariness in defining the gravitational actions which leave the classical equations of motion unchanged. A class of gravitational actions which is a functional of the metric on the boundary does not change the classical equations of motion even if they are added to the original action. Though this does not affect the equations of motion, this does bring about an arbitrariness of the energy and angular momentum in their definitions. We follow the formulation of quasilocal energy and conserved charges in Refs.[6, 7, 8]. Under the definitions of mass and angular momentum in the asymptotically anti-de Sitter spacetime, they are not constants but functions of positions in general. In our previous paper [3], we showed a general solution to the Einstein-Maxwell equations with a negative cosomological constant with one of the coordinate conditions by which all the metric components are expressed in terms of the radial coordinate variable r. This turns out to define the perimeter length of a circle with t =const. and r=const. The coordinate condition thus determines the spacetime geometry. In our previous solution there is a horizon, but the coordinate conditions which we adopted there lead to the divergence of an angular momentum both at the horizon and at the infinity. Therefore it is not a black hole solution. However this does not mean that the general solutions with SD(ASD) electromagnetic fields always have the divergence somewhere on or outside of the horizon, because we showed only one of infinite possible coordinate conditions. In our previous paper we neglected the contribution from the boundary which does not change the classical equations of motion but can contribute to the energy and charges in general. We should investigate these possibilities. The present paper is constructed as follows. We start with the summary of our general solution. We then define the definitions of energy, mass and angular momentum following Refs.[6, 7, 8]. We discuss some of the possible coordinate conditions. In the coordinate condition we adopted in the previous paper there appeared a logarithmic divergence in the angular momentum. We discuss the regularization of the divergence by the counter term emerging from the action defined on the boundary. We then discuss other coordinate conditions which suppress the divergence of the angular momentum even when $(j_0)_\phi$ in [6, 7, 8] is taken to be zero.

We shall obtain the solutions by assuming the axisymmetry with nonzero electric and

magnetic fields between which we assume a self dual(SD) (anti-self dual(ASD)) relation. The Einstein-Maxwell action is

$$S = \frac{1}{16\pi G}\int \sqrt{-g}(R - 2\Lambda - 4\pi G F^2) d^3x - S^0, \tag{1}$$

where G is Newton's constant, Λ is the negative cosmological constant and $F^2 \equiv g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}$. Here S^0 is an action which is a functional of the metric and their derivatives defined on the boundary. The Einstein equation is given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the electromagnetic field:

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu\sigma}g^{\rho\sigma} - \frac{1}{4}g_{\mu\nu}F^2. \tag{3}$$

The Maxwell equation is given by

$$\partial_\rho(\sqrt{-g}g^{\mu\nu}g^{\rho\sigma}F_{\nu\sigma}) = 0. \tag{4}$$

We use the line element given by

$$ds^2 = -N^2dt^2 + L^{-2}dr^2 + K^2(N^\phi dt + d\phi)^2, \tag{5}$$

where N, L, K and N^ϕ are functions of r only.

Since we have two Killing vectors $\partial/\partial t$ and $\partial/\partial\phi$ in the stationary axisymmetric spacetime, we can solve the Maxwell equation (4) as follows:

$$\begin{aligned} E_{\hat{r}} \equiv F_{\hat{t}\hat{r}} &= \frac{C_1}{K}, & (6)\\ B_{\hat{}} \equiv F_{\hat{r}\hat{\phi}} &= \frac{C_1N^\phi + C_2}{N}, & (7)\end{aligned}$$

where C_1 and C_2 are constants of integration. We impose the SD(ASD) equation on the electric and magnetic fields:

$$E_{\hat{r}} = \varepsilon B_{\hat{}}, \tag{8}$$

where ε takes 1 or -1 corresponding to SD or ASD equation, respectively.

As far as we do not have the asymptotically flat spacetime, we can not define mass and angular momentum as those observed at spatial infinity. We follow the discussion of qusilocal energy and conserved charges in Refs.[6, 7, 8]. The energy E, angular momentum J and mass M are expressed in terms of the metric components as

$$\begin{aligned} E &= -2LK' - 2\pi K\varepsilon_0, & (9)\\ J &= \frac{K^3L(N^\phi)'}{N} - 2\pi K(j_0)_\phi, & (10)\\ M &= EN - JN^\phi, & (11)\end{aligned}$$

where ε_0 and $(j_0)_\phi$ are the energy density and angular momentum density derived from S^0. Primes denote the derivative with respect to r. Note that, at spatial infinity, mass

and energy become identical with each other only when N and N^ϕ should satisfy $N = 1$ and $N^\phi = 0$, respectively, and when we have a finite angular momentum there. As for the explicit expressions of ε_0 and $(j_0)_\phi$ we will discuss them later in connection with the divergence of the angular momentum.

The solutions for the metric components are given by

$$L = |\Lambda|^{\frac{1}{2}} \frac{\rho^2 - r_0^2}{\rho\rho'}, \tag{12}$$

$$N = C|\Lambda|^{\frac{1}{2}} \frac{\rho^2 - r_0^2}{K}, \tag{13}$$

$$\tilde{N}^\phi = \varepsilon C|\Lambda|^{\frac{1}{2}} \frac{\rho^2 - r_0^2}{K^2}, \tag{14}$$

where $\tilde{N}^\phi$ is related to N^ϕ by

$$\tilde{N}^\phi \equiv N^\phi + C_2/C_1. \tag{15}$$

Here ρ is a function of r and related to K as

$$K^2 = \rho^2 + r_0^2 \ln|\frac{\rho^2 - r_0^2}{r_0^2}| + q, \tag{16}$$

where q is a constant and $r_0^2 \equiv 4\pi G C_1^2/|\Lambda|$. In the above equations, we can set $C = 1$ without loss of generality and so we shall adopt it henceforth. We shall next determine the constant C_1 by the surface integral of the electric field $E_{\hat{r}}$. We identify it as the electric charge:$C_1 = Q_e$. As a boundary condition, we assume that

$$N^\phi(\infty) = 0. \tag{17}$$

When $\rho^2 \to \infty$ as $r \to \infty$, we can determine C_2 by using the boundary condition as

$$C_2 = \varepsilon Q_e |\Lambda|^{\frac{1}{2}}. \tag{18}$$

The SD(ASD) equation (8) renders Eqs.(6) and (7) into

$$E_{\hat{r}} = \varepsilon B_{\hat{}} = \frac{Q_e}{K}. \tag{19}$$

So far we have not specified the explicit form of the function $\rho(r)$. We shall call its explicit specification the coordinate condition. Of a lot of coordinate conditions we shall discuss some important cases.

(i)$K = r$ case

The most naive coordinate condition that $K = r$ together with $L = N$ leads to the line element of BTZ solution [1, 2] and so we can not allow for neither electric nor magnetic field. This condition is too stringent to impose.

(ii)$\rho = r$ case

We shall discuss the coordinate condition, which we adopted in the previous paper. The condition is to set $\rho = r$ in the solutions. This yields

$$K^2 = r^2 + r_0^2 \ln|\frac{r^2 - r_0^2}{r_0^2}| + q. \tag{20}$$

Substituting the solutions and using the coordinate condition, we obtain the energy, the angular momentum and the mass as

$$
\begin{aligned}
E &= -2|\Lambda|^{\frac{1}{2}}\frac{r^2}{\sqrt{r^2+r_0^2\ln|\frac{r^2-r_0^2}{r_0^2}|+q}}-2\pi K\varepsilon_0, && (21)\\
J &= \varepsilon\frac{8\pi GQ_e^2}{|\Lambda|^{\frac{1}{2}}}(\ln|\frac{r^2-r_0^2}{r_0^2}|+\frac{q}{r_0^2})-2\pi K(j_0)_\phi, && (22)\\
M &= 2|\Lambda|\Big\{-r^2(r^2-r_0^2)+(r_0^2\ln|\frac{r^2-r_0^2}{r_0^2}|+q)(r_0^2+r_0^2\ln|\frac{r^2-r_0^2}{r_0^2}|+q)\Big\}/K^2\\
&- 2\pi|\Lambda|^{\frac{1}{2}}\{(r^2-r_0^2)\varepsilon_0+\varepsilon\frac{r_0^2+r_0^2\ln|\frac{r^2-r_0^2}{r_0^2}|+q}{K}(j_0)_\phi\}. && (23)
\end{aligned}
$$

We note that the conserved charges, mass and angualr momentum, are not constants but r dependent. The angular momentum has a logarithmic divergence as r goes to infinity if we set $(j_0)_\phi=0$. The mass has a quadratic divergence at spatial infinity while the energy has a linear divergence for $(j_0)_\phi=0$ and $\varepsilon_0=0$. Due to the logarithmic divergence of the angular momentum our solution could not be called the black hole solution.

In our previous paper we neglected the contribution from the boundary and set $(j_0)_\phi=0$ in obtaining the angular momentum. However in this paper we take the contribution from the boundary into account in determining the above quantities. As far as we do not have the criterion of defining the functional form of ε_0 and $(j_0)_\phi$, the above quantities can be put into any functional form by an appropriate choice of them. In order to avoid the arbitrariness, we adopt a criterion in the choice of them following the regularization procedure in the quntum field theory. In QED, the logarithmic divergence of mass operator is cancelled by the counter term leaving the finite part unchanged. We set the ansatz that ε_0 and $(j_0)_\phi$ should be chosen so that they cancell the logarithmically divergent parts in angular mometum and mass, if any. In the present solution we have a logarithmically divergent angular momentum at spatial infinity if $(j_0)_\phi=0$. Following the criterion, we assume that $(j_0)_\phi$ is a function of r which eliminates the logarithmic divergence as a counter term. We thus obtain the regularized angular momentum as

$$
J=-\varepsilon\frac{8\pi GQ_e^2}{|\Lambda|^{\frac{1}{2}}}\Big(1-\ln|\frac{r^2-r_0^2}{r^2-r_\infty^2}|\Big), \tag{24}
$$

where r_∞ is a constant. The regularized angular momentum is still r dependent but free from the logarithmic divergence at spatial infinity. However the divergence on the horizon($r=r_0$) still exists. If we require that $r_0^2=r_\infty^2$, we can remove the divergence also and we are left with the constant angular momentum:

$$
J=-\varepsilon\frac{8\pi GQ_e^2}{|\Lambda|^{\frac{1}{2}}}, \tag{25}
$$

which may be compared with the simultaneous regularization of the infrared and ultraviolet divergence in the quantum field theory. In a similar way we can cancel the divergences

of energy and mass by adjusting ε_0. With such regularized quantities the solution has a horizon and so we may call this the black hole solution.

(iii)$g_{\phi\phi}(\infty) \neq r^2$ case

In the case (ii) we found that the non-trivial $(j_0)_\phi$ as a counter term could lead to a black hole solution. We shall first clarify why we have such a logarithmic divergence of angular momentum for the $(j_0)_\phi = 0$ case. When we do not take the contribution from the boundary into account, we should discard the condition that $g_{\phi\phi}(= K^2)$ should approach the radial coordinate squared as far as we want the finite angular momentum.

In order to make the point clearer, we shall rewrite the Eqs.(12)-(16). We introduce x by

$$x = \frac{\rho^2 - r_0^2}{r_0^2}. \tag{26}$$

Then the metric components read

$$\begin{aligned} L &= 2|\Lambda|^{\frac{1}{2}}/(\ln|x|)', && (27)\\ N &= |\Lambda|^{\frac{1}{2}} r_0^2 x/K, && (28)\\ N^\phi &= -\varepsilon|\Lambda|^{\frac{1}{2}}\frac{r_0^2}{K^2}(1+\ln|x|+\frac{q}{r_0^2}), && (29) \end{aligned}$$

where K^2 is given by

$$K^2 = r_0^2(1 + x + \ln|x| + \frac{q}{r_0^2}). \tag{30}$$

To impose the boundary condition that $g_{\phi\phi}(= K^2)$ should approach r^2 as r becomes infinite implies that x approaches infinity as is seen from this equation. Then this causes the logarithmic divergence of the angular momentum, which behaves as $\ln x$ at infinity, in the $(j_0)_\phi = 0$ case. Therefore we need to discard the condition that $g_{\phi\phi}$ should approach the radial coordinate squared if we want the finite angular momentum with zero $(j_0)_\phi$. The origin of the divergence of angular momentum can be considered to be the nonzero matter momentum density: "Poynting pseudovector" of the electromagnetic fields (19).

In order to obtain the black hole solution without any divergence of angular momentum, we require firstly that there should be a horizon and secondly that the angular momentum is finite all through the region from the horizon to infinity. The finiteness of mass shall be checked after finding the solutions satisfying the above conditions. In order to realize the above requirements we should find the coordinate condition $x = g(r)$ which satisfy

$$\begin{aligned} &(I) \quad L \sim g(a)/g'(a) = 0, \quad \text{at} \quad r = a,\\ &(II) \quad J \sim |\ln g(r)| < \infty, \quad \text{for} \quad a \leq r < \infty. \end{aligned}$$

Here $r = a$ is the position of horizon. The previous solution does not match the conditions. Since there is an infinite class of solutions satisfying the conditions, we can not exhaust them all, but we shall illustrate one of the class of solutions. Consider $g(r)$ defined by

$$\frac{g(r)}{g'(r)} = A\sqrt{r-a}\,e^{B(r-a)}, \tag{31}$$

where A and B are positive constants. This is easily integrated to give

$$g(r) = e^{-\frac{2}{A\sqrt{B}}\mathrm{Erfc}\left(\sqrt{B(r-a)}\right)}. \tag{32}$$

Here Erfc(x) is the error function defined by

$$\mathrm{Erfc}(x) = \int_x^\infty e^{-t^2} dt, \tag{33}$$

which is a monotonically decreasing function varying from $\sqrt{\pi}/2$ to 0 as x changes from 0 to ∞. By using the solution the coordinate condition is explicitly given by

$$\rho^2 = r_0^2 \big\{ 1 + e^{-\frac{2}{A\sqrt{B}}\mathrm{Erfc}\left(\sqrt{B(r-a)}\right)} \big\}. \tag{34}$$

By using the boundary condition(17), we obtain $q/r_0^2 = -1$. The metric components are now written as

$$\begin{aligned} L &= 2|\Lambda|^{\frac{1}{2}} A\sqrt{r-a} e^{B(r-a)}, & (35)\\ N &= |\Lambda|^{\frac{1}{2}} r_0^2 g(r)/K, & (36)\\ N^\phi &= -\varepsilon|\Lambda|^{\frac{1}{2}} r_0^2 \frac{\ln|g(r)|}{K^2}, & (37) \end{aligned}$$

where K^2 is given by

$$K^2 = r_0^2 (g(r) + \ln|g(r)|). \tag{38}$$

The solution shows a peculiar topology of the spacetime. K^2 approaches r_0^2 as r goes to infinity. As is seen from Eq.(35) and Eq.(36), L^2 vanishes at $r = a$, but N^2 is finite there. The angular momentum reads

$$J = -\varepsilon \frac{8\pi G Q_e^2}{|\Lambda|^{\frac{1}{2}}} \big\{ \frac{2}{A\sqrt{B}} \mathrm{Erfc}\left(\sqrt{B(r-a)}\right) + 1 \big\}, \tag{39}$$

where we have set $(j_0)_\phi = 0$. As far as the constants A and B satisfy the condition:

$$e^{-\frac{1}{A}\sqrt{\frac{\pi}{B}}} - \frac{1}{A}\sqrt{\frac{\pi}{B}} \geq 0, \tag{40}$$

K^2 is kept positve for $a \leq r < \infty$ and the metric components are finite there. Though a further physical study is necessary till we can conclude that this is really a black hole solution, the peculiar spacetime is intriguing enough for further investigation.

We have seen how the logarithmic divergence of the angular momentum is regularized by taking the contribution from the boundary into account. If the relation $r_0 = r_\infty$ between two parameters r_0 and r_∞ should hold, we have the constant angular momentum and so we have a charged black hole solution. Even when we neglect the effect of the boundary, there is a peculiar solution with a finite angular momentum. We showed the close relation between the behavior of $g_{\phi\phi}$ and angular momentum at spatial infinity. The finiteness of the angular momentum is related to the fact that $g_{\phi\phi}$ approaches constant at the spatial infinity. We shall discuss the physical aspects of these solutions in the forthcoming paper.

References

[1] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **69** (1992) 1849.

[2] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D **48** (1993) 1506.

[3] M.Kamata and T.Koikawa, Phys.Lett B.**353** (1995) 196.

[4] G.Clément, Class.Quantum.Grav.**10**(1993) L49.

[5] K.C.K.Chan, preprint WATPHYS-TH95-05(gr-qc/9509032).

[6] J.W.York, Phys.Rev.D**33** (1986) 2092.

[7] J.D.Brown and J.W.York,Jr., Phys.Rev.D**47** (1993) 1407.

[8] J.D.Brown, J.Creighton and R.B.Mann, Phys.Rev.D**50** (1994) 6394.